PHYC 511
Spring 2019

Lecture 1 — 01/14/2019

Review of Electrostatics

Equations:

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0} & * \\ \vec{\nabla}\times\vec{E} = 0 & ** \end{cases}$$

ρ: volume charge density

From the second equation $**$, we find:

$$\vec{E} = -\vec{\nabla}\Phi$$

Φ: electric potential

This gives rise to Poisson's equation:

$$\nabla^2\Phi = \frac{-\rho}{\varepsilon_0}$$

In unbounded space, the solution to this equation is:

$$\Phi(\vec{x}) = \int \frac{\rho(\vec{x}')\, d\vec{x}'}{4\pi\varepsilon_0 |\vec{x}-\vec{x}'|}$$

$d\vec{x}'$ ⇝ volume element (also $d\tau'$)

For N point charges at $\vec{x}_1, \ldots, \vec{x}_N$, $\rho(\vec{x}) = \sum_{i=1}^{N} q_i\, \delta(\vec{x}' - \vec{x}_i)$, and hence:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{x}-\vec{x}_i|}$$

The first equation $*$ results in Gauss's law:

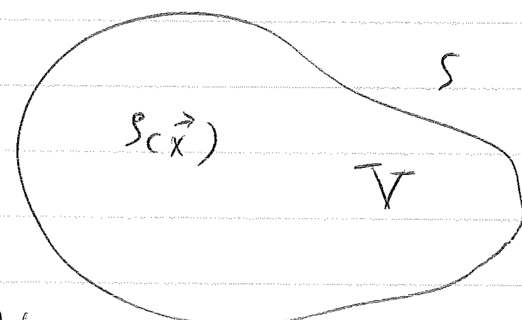
$$\oint_S \vec{E}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{E}\, d\tau = \frac{1}{\varepsilon_0}\int_V \rho\, d\tau = \frac{Q_{enc}}{\varepsilon_0}$$

Here, Q_{enc} is the total charge enclosed in the volume V by S.

Gauss's law is most useful when symmetries of the problem permit its use in calculating the electric field $\vec{E}$ directly.

Poisson Equation in Bounded Space

Consider a finite volume V bounded by a surface S:



In this case, arriving at a single solution requires pre-specified boundary conditions on S. Such a well posed problem is obtained for a given $\rho(\vec{x})$ distribution provided that either:

(1) the potential Φ is specified on S, i.e., $\Phi|_S$ is given, or

(2) the normal derivative $\frac{\partial \Phi}{\partial n}\Big|_S$ is given on S, or

(3) the mixed condition $\left(\alpha\Phi + \beta\frac{\partial\Phi}{\partial n}\right)\Big|_S$ is given on S.

These problems, called <u>Dirichlet</u>, <u>Neumann</u>, and <u>mixed</u> problems respectively, have a unique solution.

Work By Electrostatic Field

$$\Delta W_e = \int_1^2 q\vec{E}\cdot d\vec{\ell} = -q\int_1^2 \vec{\nabla}\Phi\cdot d\vec{\ell} = -q\,\Phi\Big|_1^2 = -q\,\Delta\Phi = -\Delta \underbrace{U_e}_{\text{electrostatic potential}}$$

For static problems kinetic energy is zero.

Hence, from the work-energy theorem, we have:

$$\Delta K = \Delta W_e + \underbrace{\Delta W_{non\text{-}el}}_{\text{work by non-electrostatic forces}} = 0 \Rightarrow -\Delta U_e + \Delta W_{non\text{-}el} = 0 \Rightarrow$$

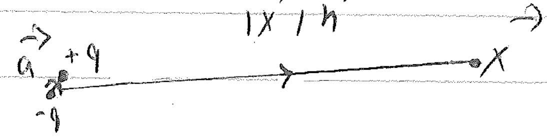
$$\Delta U_e = \Delta W_{non\text{-}el}$$

Elementary Charge Configurations

(1) Point Charge q at $\vec{x}'$:

$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0 |\vec{x}-\vec{x}'|}, \quad \vec{E}(\vec{x}) = \frac{q(\vec{x}-\vec{x}')}{4\pi\varepsilon_0 |\vec{x}-\vec{x}'|^3}$$

(2) Point dipole at the origin:



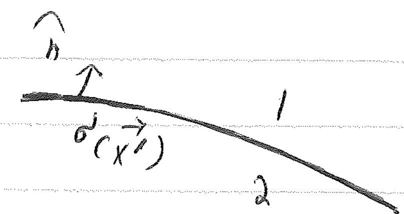
$\vec{p} = q\vec{a}$ ($q\to\infty$ & $|\vec{a}|\to 0$, $q|\vec{a}|$ fixed)

$$\Phi(\vec{x}) = \frac{\vec{p}\cdot\vec{x}}{4\pi\varepsilon_0 |\vec{x}|^3}$$

$$\vec{E} = -\vec{\nabla}\Phi = \frac{1}{4\pi\varepsilon_0}\frac{[3(\vec{p}\cdot\hat{n})\hat{n} - \vec{p}]}{|\vec{x}|^3} \qquad \hat{n} = \frac{\vec{x}}{|\vec{x}|}$$

The expression is not valid at $\vec{x}=0$ since there is a δ-function term involved (which we will consider later).

(3) Charge layer:



Non-zero surface charge density

$$\sigma = \frac{\text{charge on the surface}}{\text{area of the surface}}$$

At the surface, we have:

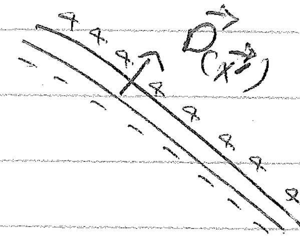
$$\Delta E_n = E_{1n} - E_{2n} = \frac{\sigma}{\varepsilon_0}$$

However, potential is continuous at the surface ($\Phi_1 = \Phi_2$) in order to have finite electric field.

(4) Dipole layer:

Non-zero surface dipole density $D = \dfrac{\text{dipole moment on the surface}}{\text{area of the surface}}$

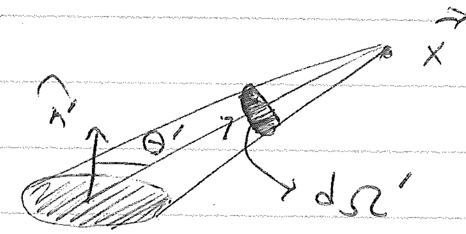
$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\vec{D}(\vec{x}')\, da' \cdot (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}$$



For a uniform dipole layer $|\vec{D}|$ is constant, and hence:

$$\Phi(\vec{x}) = \frac{D}{4\pi\varepsilon_0} \int \frac{\hat{n}' \cdot (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}\, da'$$

$$\frac{\hat{n}' \cdot (\vec{x} - \vec{x}')\, da'}{|\vec{x} - \vec{x}'|^3} = \frac{\cos\theta'\, da'}{|\vec{x} - \vec{x}'|^2} = d\Omega'$$ ← solid angle subtended by S at observation point $\vec{x}$



Therefore:

$$\Phi(\vec{x}) = \frac{D}{4\pi\varepsilon_0} \Delta\Omega$$

Note, however, that for $\vec{x}$ below the surface, we have:

$$\Phi(\vec{x}) = \frac{D}{4\pi\varepsilon_0} (-\Delta\Omega)$$

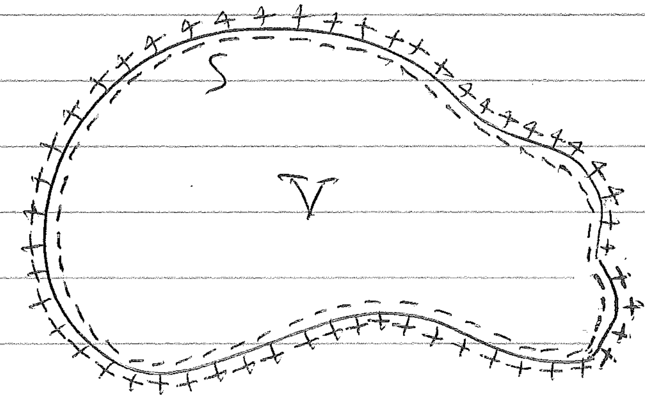
This is because $\cos\theta'$ in this case is negative.

As a result, at the surface, we have:

$$\Phi|_{S+} - \Phi|_{S-} = \frac{D}{2\pi\varepsilon_0} \Delta\Omega = \frac{D}{\varepsilon_0} \qquad (\Delta\Omega = 2\pi)$$

↑ Φ <u>discontinuous</u> at a dipole layer

Now, let us consider a closed dipole layer with uniform dipole density D. In this case:

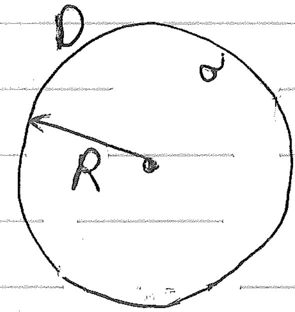


$$\Phi(\vec{x}) = \begin{cases} -\dfrac{D}{4\pi\varepsilon_0} 4\pi = \dfrac{-D}{\varepsilon_0} & \vec{x} \in V \\ 0 & \vec{x} \notin V \end{cases}$$

We note that if $\vec{x}$ is outside V, then the solid angle contributions will exactly cancel out.

This result implies that the potential is constant both inside and outside of a closed dipole layer surface, and hence $\vec{E} = 0$ everywhere except on the surface where $|\vec{E}|$ is infinitely large because of the discontinuity in Φ.

An interesting consequence is that a combination of a charge layer and a dipole layer can be used to design field and potential configurations of interest. For example, a spherical shell with a uniform charge distribution and a uniform dipole-layer density can give rise to $\Phi = 0$ both inside and at ∞ outside.



Inside, we have:

$$\Phi(\vec{x}) = \frac{-D}{\varepsilon_0} + \frac{R\sigma}{\varepsilon_0}$$

$$\Phi = 0 \Rightarrow \boxed{\sigma = \frac{D}{R}}$$

While, the potential outside is:

$$\boxed{\Phi(\vec{x}) = \frac{R^2\sigma}{\varepsilon_0 r}}$$